A UNIQUENESS RESULT ON BOUNDARY INTERPOLATION

VLADIMIR BOLOTNIKOV

Abstract. Let f be an analytic function mapping the unit disk $\mathbb{D}$ to itself. We give necessary and sufficient conditions on the local behavior of f near a finite set of boundary points that requires f to be a finite Blaschke product.

1. Introduction

The following boundary uniqueness result was presented in [3] as an intermediate step to obtain a similar result in the multivariable setting of the unit ball.

Theorem 1.1. *Let $f \in \mathcal{S}$ and let $f(z) = z + O((z-1)^4)$ as $z \to 1$. Then $f(z) \equiv z$.*

Here and in what follows, $\mathcal{S}$ denotes the Schur class of functions analytic and bounded by one in modulus on the unit disk $\mathbb{D}$. In [7], Theorem 1.1 was generalized in the following way.

Theorem 1.2. *Let $f \in \mathcal{S}$ and let b be a finite Blaschke product. Let τ be a unimodular number and let $A_{b,\tau} = b^{-1}(\tau) = \{t_1, \dots, t_d\}$ (since b is a finite Blaschke product, $A_{b,\tau}$ is a finite subset of the unit circle $\mathbb{T}$). If*

1. *$f(z) = b(z) + O((z-t_1)^4)$ as $z \to t_1$ and*
2. *$f(z) = b(z) + O((z-t_i)^{\ell_i})$ for some $\ell_i \geq 2$ as $z \to t_i$ for $i = 2, \dots, d$,*

then $f(z) \equiv b(z)$ on $\mathbb{D}$.

Thus, conditions in Theorem 1.2 are sufficient to guarantee $f(z) \equiv b(z)$. The question raised in [7] was to find necessary (in a sense) and sufficient conditions. The answer is given below. For a given real x, $[x]$ denotes the largest integer that does not exceed x.

Theorem 1.3. *Let $f \in \mathcal{S}$ and let b be a finite Blaschke product of degree d. Let $t_1, \dots, t_n$ be points on $\mathbb{T}$ and let*

$$f(z) = b(z) + o((z-t_i)^{m_i}) \quad \text{for} \quad i = 1, \dots, n \tag{1.1}$$

as z tends to t_i nontangentially and where $m_1, \dots, m_n$ are positive integers. If

$$\left[\frac{m_1+1}{2}\right] + \dots + \left[\frac{m_n+1}{2}\right] > d = \deg b, \tag{1.2}$$

then $f(z) \equiv b(z)$ on $\mathbb{D}$. Otherwise, the uniqueness result fails.

1991 *Mathematics Subject Classification.* 47A57.

Key words and phrases. Schur functions, finite Blaschke products, boundary interpolation problem.

In other words, the points $t_i \in \mathbb{T}$ can be chosen arbitrarily (regardless b) as well as degrees of convergence. To derive Theorem 1.2 from Theorem 1.3, note that if $\deg b = d$, the set $A_{b,\tau}$ consists of exactly d points on $\mathbb{T}$. The assumptions in Theorem 1.2 mean that (1.1) holds for $m_1 = 3$ and $m_i \geq 1$ for $i = 2, \ldots, d$. Then the sum on the left hand side in (1.2) is not less than $2 + (d-1) = d+1$ which is greater than d and the result follows. The proof of Theorem 1.3 is given in Section 4. It relies on some recent results on boundary interpolation [3] that we recall in Section 2 and Section 3.

2. Boundary Schwarz-Pick matrices

Let w be a Schur function. Then for every choice of $n \in \mathbb{N}$ and of n-tuples $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{D}^n$ and $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, the *Schwarz-Pick matrix* $P^w_{\mathbf{k}}(\mathbf{z})$ defined as

$$P^w_{\mathbf{k}}(\mathbf{z}) = \left[P^w_{k_i,k_j}(z_i, z_j)\right]_{i,j=1}^n \tag{2.1}$$

where

$$P^w_{k_i,k_j}(z_i, z_j) = \left[\left.\frac{1}{\ell!r!}\,\frac{\partial^{\ell+r}}{\partial z^\ell \partial\bar{\zeta}^r}\,\frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}}\right|_{\substack{z = z_i \\ \zeta = \bar{z}_j}}\right]_{\ell=0,\ldots,k_i-1}^{r=0,\ldots,k_j-1}, \tag{2.2}$$

is positive semidefinite. Indeed, every Schur function w admits a de Branges–Rovnyak realization

$$w(z) = w(0) + zC(I_{\mathcal{H}} - zA)^{-1}B \quad (z \in \mathbb{D}), \tag{2.3}$$

(see [5]) with an operator A acting on an auxiliary Hilbert space $\mathcal{H}$ and operators $B : \mathbb{C} \to \mathcal{H}$ and $C : \mathcal{H} \to \mathbb{C}$ such that the block operator $\mathbf{U} = \begin{bmatrix} A & B \\ C & w(0) \end{bmatrix}$ is a coisometry on $\mathcal{H} \oplus \mathbb{C}$ (if $\mathbf{U}$ is unitary, representation (2.3) is called a *unitary realization* of w). A consequence of equality $\mathbf{U}\mathbf{U}^* = I$ is that

$$\frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}} = C(I - zA)^{-1}(I - \bar{\zeta}A^*)^{-1}C^*.$$

Differentiating both parts in the latter identity gives

$$\frac{1}{\ell!r!}\,\frac{\partial^{\ell+r}}{\partial z^\ell \partial\bar{\zeta}^r}\,\frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}} = CA^\ell(I - zA)^{-\ell-1}(I - \bar{\zeta}A^*)^{-r-1}A^{*r}C^*$$

which allows us to represent the matrix in (2.1) as

$$P^w_{\mathbf{k}}(\mathbf{z}) = R_{\mathbf{k}}(\mathbf{z})R_{\mathbf{k}}(\mathbf{z})^*, \tag{2.4}$$

where

$$R_{\mathbf{k}}(\mathbf{z}) = \begin{bmatrix} R_{k_1}(z_1) \\ \vdots \\ R_{k_n}(z_n) \end{bmatrix} \quad \text{and} \quad R_{k_i}(z_i) = \begin{bmatrix} C(I - z_iA)^{-1} \\ CA(I - z_iA)^{-2} \\ \vdots \\ CA^{k_i-1}(I - z_iA)^{-k_i} \end{bmatrix}, \tag{2.5}$$

and to conclude that $P^w_{\mathbf{k}}(\mathbf{z}) \geq 0$. In case when w is a finite Blaschke product, the above realization arguments are more informative. In what follows, we will write $\mathcal{BF}$ for the class of all finite Blaschke products and more specifically, $\mathcal{BF}_d$ for the

set of all Blaschke products of degree d. The symbol $\mathrm{Dom}(w)$ will stand for the domain of holomorphy of w. Apperently, the next result is well known.

Lemma 2.1. *Let $w \in \mathcal{BF}_d$ and let $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$. Then*

1. *The function $P^w_{\mathbf{k}}(\mathbf{z})$ defined on $\mathbb{D}^n$ by formulas* (2.1) *and* (2.2)*, can be extended continuously to* $(\mathrm{Dom}(w))^n$.
2. *For every $\mathbf{z} \in (\mathrm{Dom}(w))^n$, the matrix $P^w_{\mathbf{k}}(\mathbf{z})$ is positive semidefinite and* $\operatorname{rank} P^w_{\mathbf{k}}(\mathbf{z}) = \min\{|\mathbf{k}|, d\}$ *where we have set $|\mathbf{k}| := k_1 + \dots + k_n$.*

Proof: Since w is a rational function of McMillan degree d, it admits ([1, Chapter 4]) a minimal realization

$$w(z) = w(0) + zC(I_d - zA)^{-1}B \quad (z \in \mathrm{Dom}(w)), \tag{2.6}$$

holding for all $z \in \mathrm{Dom}(w)$, with $\mathcal{H} = \mathbb{C}^d$ and matrices $A \in \mathbb{C}^{d\times d}$, $B \in \mathbb{C}^{d\times 1}$, $C \in \mathbb{C}^{1\times d}$ such that

$$\bigcap_{j=0}^{d-1} \operatorname{Ker} CA^j = \{0\} \quad \text{and} \quad \det(I - zA) \neq 0 \ \ (z \in \mathrm{Dom}(w)). \tag{2.7}$$

Furthermore, if w inner, then the matrices A, B and C can be chosen so that the minimal realization (2.6) will be unitary [4]. The same result comes out of the de Branges–Rovnyak model: if w is inner, the de Branges–Rovnyak realization is unitary (not just coisometric) with the state space $\mathcal{H} = H^2 \ominus wH^2$; if $w \in \mathcal{BF}_d$, then $\dim \mathcal{H} = d$ and (2.6) is obtained upon identifying $\mathcal{H}$ with $\mathbb{C}^d$.

Since realization (2.6) is unitary, formulas (2.4) and (2.5) hold. By (2.5), $R_{\mathbf{k}}(\mathbf{z})$ is analytic on (more precisely, can be extended analytically to) $(\mathrm{Dom}(w))^n$ and then formula (2.4) gives the desired extension of $P^w_{\mathbf{k}}(\mathbf{z})$ to the all of $(\mathrm{Dom}(w))^n$. By (2.5), $R_{\mathbf{k}}(\mathbf{z}) \in \mathbb{C}^{|\mathbf{k}|\times d}$, and therefore we have from (2.4)

$$P^w_{\mathbf{k}}(\mathbf{z}) \geq 0 \quad \text{and} \quad \operatorname{rank} P^w_{\mathbf{k}}(\mathbf{z}) \leq \min\{|\mathbf{k}|, d\}. \tag{2.8}$$

On the other hand, if $|\mathbf{k}| = d$, the square matrix $R_{\mathbf{k}}(\mathbf{z})$ is not singular. Indeed, assuming that it is singular, we take a nonzero vector $x \in \mathbb{C}^d$ such that

$$R_{\mathbf{k}}(\mathbf{z}) \prod_{j=1}^{n} (I - z_j A)^{k_j} x = 0.$$

By (2.5), the latter matrix equation reduces to the system of $d = |\mathbf{k}|$ equalities

$$CA^{\ell}(I - z_i A)^{k_i - \ell - 1} \prod_{j \neq i} (I - z_j A)^{k_j} x = 0$$

for $\ell = 0, \dots, k_i - 1$ and $i = 1, \dots, n$. Expanding polynomials leads to a homogeneous liner system (with respect to Cx, CAx, $\dots$ $CA^{d-1}x$) with the nonzero Vandermonde-like determinant from which it follows that $GA^{\ell}x = 0$ for $\ell = 0, \dots, d-1$. Then $x = 0$, by the first relation in (2.7), and thus, $\det R_{\mathbf{k}}(\mathbf{z}) \neq 0$. By (2.4), $P^w_{\mathbf{k}}(\mathbf{z}) > 0$ whenever $\mathbf{z} \in (\mathrm{Dom}(w))^n$ and $|\mathbf{k}| = d$. Finally if $\mathbf{k} = (k_1, \dots, k_n)$ is any tuple with $|\mathbf{k}| = \widetilde{d} < d$, let $\widetilde{\mathbf{k}} = (k_1, \dots, k_{n-1}, k_n + d - \widetilde{d})$ so that $|\widetilde{\mathbf{k}}| = d$. Since $P^w_{\mathbf{k}}(\mathbf{z})$ is the top $\widetilde{d} \times \widetilde{d}$ principal submatrix in $P^w_{\widetilde{\mathbf{k}}}(\mathbf{z})$ and since the latter matrix is positive definite by the preceding analysis, we have

$$P^w_{\mathbf{k}}(\mathbf{z}) > 0 \quad \text{whenever} \ \ \mathbf{z} \in (\mathrm{Dom}(w))^n \ \ \text{and} \ \ |\mathbf{k}| < d. \tag{2.9}$$

Combining (2.9) and (2.8) gives the second assertion of the lemma and completes the proof. $\square$

Given $w \in \mathcal{BF}$ and a "boundary" tuple $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{T}^n$, Lemma 2.1 allows us to define the *boundary Schwarz-Pick matrix* $P_{\mathbf{k}}^w(\mathbf{t}) = R_{\mathbf{k}}(\mathbf{t})R_{\mathbf{k}}(\mathbf{t})^*$ via factorization formula (2.4) for every $\mathbf{k} \in \mathbb{N}^n$. However, we are more interested in boundary Schwarz-Pick matrices for more general Schur functions. The following definition looks appropriate:

Definition 2.2. *Given $w \in \mathcal{S}$, $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{T}^n$, the boundary Schwarz-Pick matrix is defined by*

$$P_{\mathbf{k}}^w(\mathbf{t}) = \lim_{\mathbf{z} \to \mathbf{t}} P_{\mathbf{k}}^w(\mathbf{z}) \tag{2.10}$$

as $z_i \in \mathbb{D}$ tends to t_i nontangentially for $i = 1, \dots, n$, provided the limit in (2.10) *exists.*

Here and in what follows, "the limit exists" means also that it is finite. By (2.1) and (2.2), $P_{\mathbf{k}}^w(\mathbf{t})$ is of the form

$$P_{\mathbf{k}}^w(\mathbf{t}) = \left[P_{k_i,k_j}^w(t_i, t_j)\right]_{i,j=1}^n \tag{2.11}$$

where

$$P_{k_i,k_j}^w(t_i, t_j) = \lim_{\substack{z \to t_i \\ \zeta \to t_j}} \left[\frac{1}{\ell! r!} \frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{\zeta}^r} \frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}}\right]_{\ell=0,\dots,k_i-1}^{r=0,\dots,k_j-1}. \tag{2.12}$$

A necessary and sufficient condition for the limits (2.12) to exist is that

$$\liminf_{z \to t_i} \frac{\partial^{2k_i-2}}{\partial z^{k_i-1}\partial \bar{z}^{k_i-1}} \frac{1 - |w(z)|^2}{1 - |z|^2} < \infty \quad \text{for } \ i = 1, \dots, n, \tag{2.13}$$

where $z \in \mathbb{D}$ tends to t_i arbitrarily (not necessarily nontangentially). Necessity is self-evident since the bottom diagonal entries in the diagonal blocks $P_{k_i,k_i}^w(t_i, t_i)$ are the nontangential (angular) limits

$$\lim_{z,\zeta \to t_i} \frac{1}{((k_i - 1)!)^2} \frac{\partial^{2k_i-2}}{\partial z^{k_i-1}\partial \bar{\zeta}^{k_i-1}} \frac{1 - w(z)\overline{w(\zeta)}}{1 - z\bar{\zeta}}$$

and their existence clearly implies (2.13). The sufficiency part was proved in [3] along with some other important consequences of conditions (2.13) that are recalled in the following theorem.

Theorem 2.3. *Let $t_1, \dots, t_n \in \mathbb{T}$, $k_1, \dots, k_n \in \mathbb{N}$, $w \in \mathcal{S}$ and let us assume that conditions* (2.13) *are met. Then*

(1) *The following nontangential boundary limits exist*

$$w_j(t_i) := \lim_{z \to t_i} \frac{w^{(j)}(z)}{j!} \quad \textit{for } \ j = 0, \dots, 2k_i - 1; \ i = 1, \dots, n. \tag{2.14}$$

(2) *The nontangential boundary limit* (2.10) *exists (or equivalently all the limits in* (2.12)*) exist) and can be expressed in terms of the limits* (2.14) *as follows:*

$$P_{k_i,k_j}^w(t_i, t_j) = \mathbf{H}_{k_i,k_j}^w(t_i, t_j)\Psi_{k_j}(t_j)\mathbf{T}_{k_j}^w(t_j)^* \tag{2.15}$$

where $\Psi_{k_j}(t_j)$ is the $k_j \times k_j$ upper triangular matrix with the entries

$$\psi_{r\ell} = \begin{cases} 0, & \text{if } \ r > \ell \\ (-1)^{\ell}\binom{\ell}{r} t_0^{\ell+r+1}, & \text{if } \ r \le \ell \end{cases} \qquad (r, \ell = 0, \dots, k_j - 1), \tag{2.16}$$

where $\mathbf{T}^w_{k_j}(t_j)$ is the lower triangular Toeplitz matrix:

$$\mathbf{T}^w_{k_j}(t_j) = \begin{bmatrix} w_0(t_j) & 0 & \dots & 0 \\ w_1(t_j) & w_0(t_j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ w_{k_j-1}(t_j) & \dots & w_1(t_j) & w_0(t_j) \end{bmatrix},$$

and where $\mathbf{H}^w_{k_i,k_j}(t_i, t_j)$ is defined for $i = j$ as the Hankel matrix

$$\mathbf{H}^w_{k_i,k_i}(t_i, t_i) = \begin{bmatrix} w_1(t_i) & w_2(t_i) & \dots & w_{k_i}(t_j) \\ w_2(t_i) & w_3(t_i) & \dots & w_{k_i+1}(t_i) \\ \vdots & \vdots & & \vdots \\ w_{k_i}(t_i) & w_{k_i+1}(t_i) & \dots & w_{2k_i-1}(t_i) \end{bmatrix} \tag{2.17}$$

and entrywise (if $i \ne j$) by

$$\begin{aligned} [\mathbf{H}(t_i, t_j)]_{r,\ell} \quad &= \quad \sum_{\alpha=0}^{r} (-1)^{r-\alpha} \binom{\ell + r - \alpha}{\ell} \frac{w_\alpha(t_i)}{(t_i - t_j)^{\ell+r-\alpha+1}} \\ &\quad - \sum_{\beta=0}^{\ell} (-1)^{r} \binom{\ell + r - \beta}{r} \frac{w_\beta(t_j)}{(t_i - t_j)^{\ell+r-\beta+1}} \end{aligned} \tag{2.18}$$

for $r = 0, \dots, k_i - 1$ and $\ell = 0, \dots, k_j - 1$.

(3) *It holds that*

$$|w_0(t_i)| = 1 \quad (i = 1, \dots, n) \quad \text{and} \quad P^w_{\mathbf{k}}(\mathbf{t}) \ge 0. \tag{2.19}$$

Remark 2.4. Once the two first statements in Theorem 2.3 are proved, the third statement is immediate. Inequality $P^w_{\mathbf{k}}(\mathbf{t}) \ge 0$ follows from (2.10) and the fact that $P^w_{\mathbf{k}}(\mathbf{z}) \ge 0$ for every $z \in \mathbb{D}$. Furthermore, existence of the limits (2.12) implies in particular that the nontangential boundary limits $\lim\limits_{z \to t_i} \dfrac{1 - |w(z)|^2}{1 - |z|^2}$ exist for $i = 1, \dots, n$ (and are finite) which together with existence of the nontangential limits $w_0(t_i)$ in (2.14) implies that $|w_0(t_i)| = 1$.

Remark 2.5. In case $n = 1$ and $k_1 = 1$, Theorem 2.3 reduces to the classical Carathéodory-Julia theorem [8] on angular derivatives.

Remark 2.6. In [9], I. Kovalishina considered the single point case ($n = 1$ and $k_1 > 1$) under an additional assumption that w satisfies the symmetry relation $w(z)\overline{w(1/\bar{z})} \equiv 1$ in some neighborhood of t_1. A remarkable "Hankel-Ψ-Toeplitz" structure (2.15) of $P^w_{k_1,k_1}(t_1, t_1)$ has been observed there.

Carathéodory-Julia type conditions (2.13) are worth a formal definition.

Definition 2.7. *Given n-tuples $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{T}^n$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, a Schur function w is said to belong to the class $\mathcal{S}_{\mathbf{k}}(\mathbf{t})$ if it meets conditions (2.13).*

Statement (1) in Theorem 2.3 shows that the definition (2.10) of the boundary Schwarz-Pick matrix $P^w_{\mathbf{k}}(\mathbf{t})$ makes sense if and only if $w \in \mathcal{S}_{\mathbf{k}}(\mathbf{t})$. Statement (2) expresses $P^w_{\mathbf{k}}(\mathbf{t})$ in terms of boundary limits of w and of its derivatives. An interesting point in (2.19) is that the block matrix $P^w_{\mathbf{k}}(\mathbf{t})$ of the form (2.11) constructed via structured formulas (2.15)–(2.18) (rather than by the limits (2.12)) does not look like a Hermitian matrix and nevertheless, it turns out to be Hermitian (and even positive semidefinite) due to conditions (2.13). The next theorem (see [3] for the proof) shows that relations (2.19) are characteristic for the class $\mathcal{S}_{\mathbf{k}}(\mathbf{t})$.

Theorem 2.8. *Let w be a Schur function, let $\mathbf{t} \in \mathbb{T}^n$, $\mathbf{k} \in \mathbb{N}^n$ and let us assume that the nontangential limits* (2.14) *exist and satisfy conditions* (2.19) *where $P^w_{\mathbf{k}}(\mathbf{t})$ is the matrix constructed from the limits* (2.14) *via formulas* (2.15)–(2.18). *Then $w \in \mathcal{S}_{\mathbf{k}}(\mathbf{t})$.*

From the computational point of view, it is much easier to construct the boundary Schwarz-Pick matrix $P^w_{\mathbf{k}}(\mathbf{t})$ via formulas (2.15)–(2.18), than by (2.12) (for example, if w is a rational function, the boundary limits $w_i(t_j)$ are just the Taylor coefficients from the expansion of w around t_i). However, as follows from Theorems 2.3 and 2.8, the matrix constructed in this way will be indeed the boundary Schwarz-Pick matrix if and only if conditions (2.19) are satisfied.

3. Boundary interpolation for classes $\mathcal{S}_{\mathbf{k}}(\mathbf{t})$

The following interpolation problem has been studied in [3].

Problem 3.1. *Given $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{T}^n$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and numbers $b_{ij} \in \mathbb{C}$ ($j = 0, \ldots, k_i - 1$; $i = 1, \ldots, n$), find all functions $f \in \mathcal{S}_{\mathbf{k}}(\mathbf{t})$ such that*

$$f_j(t_i) := \lim_{z \to t_i} \frac{f^{(j)}(z)}{j!} = b_{ij} \quad (j = 0, \ldots, 2k_i - 1;\ i = 1, \ldots, n). \tag{3.1}$$

where all the limits are nontangential.

This interpolation problem makes perfect sense: if f belongs to $\mathcal{S}_{\mathbf{k}}(\mathbf{t})$, then the nontangential limits in (3.1) exist by Theorem 2.3; we just want them to be equal to the preassigned numbers. Let define the $|\mathbf{k}| \times |\mathbf{k}|$ matrix P (the *Pick matrix* of the problem) by formulas similar to (2.15)–(2.18), but with $w_j(t_i)$ replaced by b_{ij}:

$$P = [P_{ij}]_{i,j=1}^n \quad \text{with} \quad P_{ij} = H_{ij} \cdot \Psi_{k_j}(t_j) \cdot T_j^*, \tag{3.2}$$

where $\Psi_{k_j}(t_j)$ is the upper triangular matrix with the entries given in (2.16), where T_i is the lower triangular Toeplitz matrix and H_{ii} is the Hankel matrix defined by

$$T_i = \begin{bmatrix} b_{i,0} & & 0 \\ \vdots & \ddots & \\ b_{i,k_j-1} & \ldots & b_{i,0} \end{bmatrix}, \quad H_{ii} = \begin{bmatrix} b_{i,1} & \cdots & b_{i,k_i} \\ \vdots & & \vdots \\ b_{i,k_i} & \cdots & b_{i,2k_i-1} \end{bmatrix} \tag{3.3}$$

for $i = 1, \ldots, n$ and where the matrices H_{ij} (for $i \neq j$) are defined entrywise by

$$\begin{aligned} [H_{ij}]_{r,\ell} \quad = \quad & \sum_{\alpha=0}^{r} (-1)^{r-\alpha} \begin{pmatrix} \ell + r - \alpha \\ \ell \end{pmatrix} \frac{b_{i,\alpha}}{(t_i - t_j)^{\ell+r-\alpha+1}} \\ & - \sum_{\beta=0}^{\ell} (-1)^{r} \begin{pmatrix} \ell + r - \beta \\ r \end{pmatrix} \frac{b_{j,\beta}}{(t_i - t_j)^{\ell+r-\beta+1}}. \end{aligned} \tag{3.4}$$

The purpose of this construction is clear: the matrix P constructed above depends on the interpolation data only; on the other hand, for every solution f of Problem 3.1, the boundary Schwarz-Pick matrix $P^f_{\mathbf{k}}(\mathbf{t})$ must be equal to P, by the very construction.

Theorem 3.2. *Let P be the matrix defined in* (3.2)*. Then*

(1) *If Problem 3.1 has a solution, then*

$$|b_{i,0}| = 1 \quad (i = 1, \dots, n) \quad \textit{and} \quad P \geq 0. \tag{3.5}$$

(2) *If* (3.5) *holds and $P > 0$, then Problem 3.1 has infinitely many solutions.*
(3) *If* (3.5) *holds and P is singular, then Problem 3.1 has at most one solution.*
(4) *If* (3.5) *holds and f is a Schur function satisfying conditions* (3.1)*, then necessarily $f \in \mathcal{S}_{\mathbf{k}}(\mathbf{t})$.*

The first statement follows from Statement (3) in Theorem 2.3, since $b_{i,0} = f_0(t_i)$ and $P^f_{\mathbf{k}}(\mathbf{t}) = P$ for every solution f of Problem 3.1. The last statement follows from Theorem 2.8. The second statement is proved in [3] where moreover, a linear fractional parametrization of all solutions of Problem 3.1 (in case $P > 0$) is given. The third statement also was proved in [3].

The proof of Theorem 1.3 will rest on Theorem 3.2 and the following simple observation.

Proposition 3.3. *Let $\widetilde{P} = [p_{ij}]$ be an $r \times r$ Hermitian matrix and let us assume that its principal submatrix $P = [p_{i_\alpha, i_\beta}]_{\alpha,\beta=1}^{\ell}$ is positive definite. Then $\widetilde{P}$ can be turned into a positive definite matrix upon an appropriate modification of the $r-\ell$ diagonal entries p_{ii} for $i \notin \{i_1, \dots, i_\ell\}$ (we will call these entries the diagonal entries of $\widetilde{P}$ complementary to the principal submatrix P).*

Proof: Without loss of generality we can assume that P is the leading principal submatrix of $\widetilde{P}$ so that $\widetilde{P} = \begin{bmatrix} P & R^* \\ R & D \end{bmatrix}$. Let us modify the diagonal entries in D as follows:

$$\widetilde{P}' = \begin{bmatrix} P & R^* \\ R & D' \end{bmatrix} \quad \text{where } D' = D + \rho I_{r-\ell} \ \ (\rho > 0).$$

Since $P > 0$, the factorization formula

$$\begin{bmatrix} P & R^* \\ R & D' \end{bmatrix} = \begin{bmatrix} I_\ell & 0 \\ RP^{-1} & I_{r-\ell} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & D' - RP^{-1}R^* \end{bmatrix} \begin{bmatrix} I_\ell & P^{-1}R^* \\ 0 & I_{r-\ell} \end{bmatrix}$$

shows that $\widetilde{P}' > 0$ if and only if $D' - RP^{-1}R^* = \rho I_{r-\ell} + D - RP^{-1}R^* > 0$ and the latter inequality indeed can be achieved if ρ is large enough. $\square$

4. Proof of Theorem 1.3

Let us assume for a moment that the Schur function f in (1.1) is not given and let us consider the following interpolation problem.

Problem 4.1. *Given $t_1, \dots, t_n \in \mathbb{T}$, $m_1, \dots, m_n \in \mathbb{N}$ and $b \in \mathcal{BF}_d$, find all Schur functions f satisfying asymptotic equations* (1.1).

Note that conditions (1.1) can be reformulated equivalently (see e.g., [2, Corollary 7.9] for the proof) as follows: *the nontangential boundary limits $f_j(t_i)$ exist and satisfy*

$$f_j(t_i) := \lim_{z\to t_i} \frac{f^{(j)}(z)}{j!} = \frac{b^{(j)}(t_i)}{j!} =: b_{ij} \quad \textit{for } \; j=0,\dots,m_i; \; i=1,\dots,n. \tag{4.1}$$

Define the integers $k_i := \left[\frac{m_i+1}{2}\right]$ for $i=1,\dots,n$ so that $m_i = 2k_i - 1$ or $m_i = 2k_i$. Reindexing if necessary, we can assume without loss of generality that the first ℓ integers $m_1,\dots,m_\ell$ are odd while the remaining ones (if any) are even. Now we split conditions (4.1) into two parts:

$$f_j(t_i) = \frac{b^{(j)}(t_i)}{j!} =: b_{ij} \quad \text{for } \; j=0,\dots,2k_i-1; \; i=1,\dots,n \tag{4.2}$$

and

$$f_{2k_i}(t_i) = \frac{b^{(2k_i)}(t_i)}{(2k_i)!} =: b_{i,2k_i} \quad \text{for } \; i=\ell+1,\dots,n. \tag{4.3}$$

First we consider the interpolation problem with interpolation conditions (4.2) (this problem is "truncated" with respect to Problem 4.1). This problem looks like Problem 3.1; however, it is more special, since that data $\{b_{ij}\}$ comes from certain $b \in \mathcal{BF}_d$. In other words, the Pick matrix P of the problem (4.2) coincides with the boundary Schwarz-Pick matrix $P^b_{\mathbf{k}}(\mathbf{t})$. Then we may conclude by Lemma 2.1 that $P \geq 0$ and

$$\operatorname{rank} P = \min\{|\mathbf{k}|, \, d\}. \tag{4.4}$$

Thus, the second condition in (3.5) is met, while the first condition holds since $b_{i,0} = b(t_i)$ and $b \in \mathcal{BF}$. Assuming that inequality (1.2) is in force, i.e., that

$$d < \sum_{i=1}^{n} \left[\frac{m_i+1}{2}\right] = \sum_{i=1}^{n} k_i = |\mathbf{k}|$$

we conclude from (4.4) that P is singular and then by Statement (3) in Theorem 3.2, there is at most one $f \in \mathcal{S}$ satisfying conditions (4.2). Therefore (since (4.2) is just part of (4.1)), there is at most one $f \in \mathcal{S}$ satisfying conditions (4.1). A self-evident observation that the Schur function b does satisfy (4.1) (this information is contained in (4.1)) gives the desired uniqueness: there are no functions f in $\mathcal{S}$ different from b that satisfy interpolation conditions (4.1) or, equivalently, asymptotic equalities (1.1). Thus, once (1.2) is in force and f is subject to (1.1), we have necessarily $f(z) \equiv b(z)$. This completes the proof of the first statement in Theorem 1.3. It remains to show that the uniqueness result fails whenever $|\mathbf{k}| \leq d$. In this case we conclude from (4.4) that the $|\mathbf{k}| \times |\mathbf{k}|$ matrix P is positive definite and then, by Statement (2) in Theorem 3.2, there are infinitely many Schur functions f satisfying conditions (4.2). In case all m_i's are odd, this completes the proof: conditions (4.2) are identical with (4.1) and thus, there are infinitely many Schur functions satisfying asymptotic (1.1). The general case (when the set of conditions (4.3) is not empty) requires one step more.

Assuming that $|\mathbf{k}| \leq d$ so that the Pick matrix $P = P^b_{\mathbf{k}}(\mathbf{t})$ corresponding to interpolation problem (4.2) is positive definite and that $\ell < n$ in (4.3), let us attach

interpolation conditions

$$f_{2k_i+1}(t_i) = \frac{b^{(2k_i+1)}(t_i)}{(2k_i+1)!} =: b_{i,2k_i+1} \quad \text{for } \; i = \ell+1, \dots, n \tag{4.5}$$

to (4.3) and let us consider the extended interpolation problem (for Schur class functions) with interpolation conditions (4.2), (4.3) and (4.5). The collection of b_{ij}'s appearing in (4.2) and (4.3) will be called the *original data*, the collection $\{b_{i,2k_i+1}\}$ from (4.5) will be called the *supplementary data* whereas their union will be referred to as to the *extended data*.

For the extended interpolation problem we have an even number of conditions for each interpolating point t_i which allows us to construct the corresponding extended Pick matrix $\widetilde{P}$ via formulas (3.2):

$$\widetilde{P} = \left[\widetilde{P}_{ij}\right]_{i,j=1}^{n} \quad \text{where} \quad \widetilde{P}_{ij} = \widetilde{H}_{ij} \cdot \Psi_{\widetilde{k}_j}(t_j) \cdot \widetilde{T}_j^* \tag{4.6}$$

and where $\widetilde{H}_{ij}$ and $\widetilde{T}_j$ are defined by formulas (3.3), (3.4) with k_i replaced by $\widetilde{k}_i$. It is clear that $\widetilde{P}$ coincides with the boundary Schwarz-Pick matrix $P^b_{\widetilde{\mathbf{k}}}(\mathbf{t})$ based on the same $b \in \mathcal{BF}_d$, the same $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{T}^n$ and

$$\widetilde{\mathbf{k}} = (\widetilde{k}_1, \dots, \widetilde{k}_n) = (k_1, \dots, k_\ell, k_{\ell+1}+1, \dots, k_n+1) \in \mathbb{N}^n.$$

Of course, all the entries in $\widetilde{P}$ are expressed in terms of the extended data. However, it turns out that all its entries but ℓ diagonal ones are uniquely determined from the *original data*. Indeed, if $i \neq j$, then $\widetilde{H}_{ij}$ and $\widetilde{T}_j$ (and therefore, $\widetilde{P}_{ij}$) are expressed via formulas (3.3), (3.4) in terms the numbers $b_{i,0}, \dots, b_{i,\widetilde{k}_i-1}$ and $b_{j,0}, \dots, b_{j,\widetilde{k}_j-1}$ all of which are contained in the original data, since $\widetilde{k}_i - 1 \le k_i \le 2k_i - 1$.

Now we examine the diagonal blocks $\widetilde{P}_{ii}$ for $i > \ell$ (if $i \le \ell$, then $\widetilde{P}_{ii} = P_{ii}$ is completely determined by the original data). By (4.6) and (3.3),

$$\widetilde{P}_{ii} = \begin{bmatrix} b_{i,1} & \cdots & b_{i,k_i} & b_{i,k_i+1} \\ \vdots & & \vdots & \vdots \\ b_{i,k_i} & \dots & b_{i,2k_i-1} & b_{i,2k_i} \\ b_{i,k_i+1} & \dots & b_{i,2k_i} & b_{i,2k_i+1} \end{bmatrix} \Psi_{k_i+1}(t_i) \begin{bmatrix} \overline{b}_{i,0} & \dots & \overline{b}_{i,k_i} \\ & \ddots & \vdots \\ 0 & & \overline{b}_{i,0} \end{bmatrix}. \tag{4.7}$$

It is readily seen from (4.7) that the only entry in $\widetilde{P}_{ii}$ that depends on the supplementary data is the the bottom diagonal entry

$$\gamma_i := \left[\widetilde{P}_{ii}\right]_{k_i,k_i} = \begin{bmatrix} b_{i,k_i+1} & \cdots & b_{i,2k_i+1} \end{bmatrix} \Psi_{k_i+1}(t_i) \begin{bmatrix} b_{i,k_i} & \cdots & b_{i,0} \end{bmatrix}^* \tag{4.8}$$

which, on account of (2.16), can be written as

$$\begin{aligned} \gamma_i \;&=\; (-1)^{k_i} t_i^{2k_i+1} b_{i,2k_i+1} \overline{b}_{i,0} \\ &\quad + \sum_{r=0}^{k_i-1} b_{i,k_i+r+1} \sum_{j=k_i+r}^{k_i} (-1)^j t_i^{k_i+r+j+1} \begin{pmatrix} j \\ k_i+r \end{pmatrix} \overline{b}_{i,n_i-j}. \end{aligned} \tag{4.9}$$

Since $\widetilde{P}$ coincides with the boundary Schwarz-Pick matrix $P^b_{\widetilde{\mathbf{k}}}(\mathbf{t})$, it is positive semidefinite (by Lemma 2.1) and Hermitian, in particular. Furthermore, the Pick matrix $P = P^b_{\mathbf{k}}(\mathbf{t})$ of the problem (4.2) is a positive definite principal submatrix of $\widetilde{P}$. The diagonal entries in $\widetilde{P}$ complementary to P are exactly γ_i's from (4.8), the

bottom diagonal entries in the blocks $\widetilde{P}_{ii}$ of $\widetilde{P}$ for $i = \ell+1,\dots,n$. By Proposition 3.3, upon replacing γ_i in $\widetilde{P}$ by appropriately chosen (sufficiently large) positive numbers γ_i' (for $i = \ell+1,\dots,n$) and keeping all the other entries the same, one gets a positive definite matrix $\widetilde{P}'$. Furthermore, for each chosen γ_i', there exists (the unique) $b'_{i,2k_i+1}$ such that

$$\begin{aligned}\gamma_i' &= (-1)^{k_i} t_i^{2k_i+1} b'_{i,2k_i+1}\overline{b}_{i,0}\\ &\quad + \sum_{r=0}^{k_i-1} b_{i,k_i+r+1} \sum_{j=k_i+r}^{k_i} (-1)^j t_i^{k_i+r+j+1} \begin{pmatrix} j \\ k_i+r \end{pmatrix} \overline{b}_{i,n_i-j}\end{aligned}$$

(since $b_{i,0} \neq 0$, the latter equality can be solved for $b'_{i,2k_i+1}$). Now we replace the supplementary interpolation conditions (4.5) by

$$f_{2k_i+1}(t_i) = b'_{i,2k_i+1} \quad \text{for} \;\; i = \ell+1,\dots,n \tag{4.10}$$

where the numbers on the right have nothing to do with the finite Blaschke product b anymore. It is easily seen that the Pick matrix of the modified extended interpolation problem with interpolation conditions (4.2), (4.3) and (4.10) is $\widetilde{P}'$. Since it is positive definite, there are (by Statement (2) in Theorem 3.2) infinitely many Schur functions f satisfying these interpolation conditions. Thus, there are infinitely many Schur functions satisfying (4.2), (4.3) (that is, (4.1)) or equivalently, the asymptotic equalities (1.1). Thus, the uniqueness conclusion in Theorem 1.3 fails which completes the proof. □

References

[1] J.A. Ball, I. Gohberg and L. Rodman, *Interpolation of Rational Matrix Functions*, **OT45**, Birkhäuser-Verlag, Basel-Boston, 1990.

[2] V. Bolotnikov and H. Dym, *On boundary interpolation for matrix valued Schur functions*, Mem. Amer. Math. Soc., to appear.

[3] V. Bolotnikov and A. Kheifets, *A higher multiplicity analogue of the Carathéodory–Julia theorem and related boundary interpolation*, Preprint.

[4] V. Bolotnikov and L. Rodman, *Krein–Langer factorizations via pole triples*, Integral Equations and Operator Theory **47** (2003), no. 2, 169–195.

[5] L. de Branges and J. Rovnyak, *Square summable power series*, Holt, Rinehart and Winston, New York, 1966.

[6] D. Burns and S. G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary*, J. Amer. Math. Soc. **7** (1994), no. 3, 661–676.

[7] D. Chelst, *A generalized Schwarz lemma at the boundary*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3275–3278.

[8] C. Carathéodory, *Über die Winkelderivierten von beschränkten Funktionen*, Sitzungber. Preuss. Akad. Wiss. (1929), 39–52.

[9] I. V. Kovalishina, *multiple boundary interpolation problem for contractive matrix–valued functions in the unit circle*, J. Soviet Math. **52(6)** (1990), 3467–3481.

Department of Mathematics, College William and Mary, Williamsburg, Virginia 23187-8795, U. S. A.